SOME INTEGRALS OF THE EQUATIONS OF MOTION OF A DYNAMICALLY VARIABLE POINT

(NEKOTORYE INTEGRALY URAVNENII DVIZHENIIA DINAMICHESKI MENIAIUSHCHEISIA TOCHKI)

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The author shows the existence of the energy integral of the motion of a dynamically changing object (an object or body of variable mass) when the absolute velocity of the particles is colinear and equal to half the velocity of the point. Under this condition the integral curves of the equation of motion of the dynamically variable point coincide with the geodetics in the conformal Riemannian space.

1. Let us consider the motion of a dynamically variable point in a Riemannian space with the metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (\text{usually } \alpha,\ \beta = 1,\ 2,\ 3) \tag{1}$$

By means of the linear transformation

$$y^i = y^i\,(x', \ldots, x^n)$$

or directly from the dynamic form [1]

$$\Phi = m g_{\alpha\beta} \dot{x}^\beta dx^\alpha - \left\{ m\dot{x}_\alpha \dot{x}^\alpha - T - \int_1^2 (X_\alpha + X_{(r)\alpha}) dx^\alpha \right\} dt$$

the first system of equations of Pfaff for the coordinates x^α yields the covariant

$$\frac{\delta \dot{x}_\alpha}{\delta t} = \ddot{x}_\alpha + \Gamma_{\alpha,\,\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = Q_\alpha + P_\alpha \tag{2}$$

and contravariant

$$\frac{\delta \dot{x}^\alpha}{\delta t} = \ddot{x}^\alpha + \Gamma^\alpha_{\cdot\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = Q^\alpha + P^\alpha \tag{3}$$

equations of motion of the dynamically variable point. Here $\Gamma_{\alpha,\beta\gamma}$ and

$\Gamma_{\cdot\beta\gamma}$ are the Christoffel symbols of the first and second kind, while

$$Q_\alpha = \frac{1}{m} X_\alpha, \qquad P_\alpha = \frac{1}{m} X_{(r)\alpha} = \frac{1}{m}\frac{dm}{dt}(u_\alpha - \dot{x}_\alpha)$$

denote the covariant coordinates of the active and reactive forces.

2. If the point is moving in a field of conservative forces when $\mathbf{u} = 1/2\ \mathbf{v}$, Equations (2) and (3) take the forms

$$\frac{\delta \dot{x}_\alpha}{\delta t} = \frac{1}{m}\frac{\partial U}{\partial x^\alpha} - \frac{1}{2m}\frac{dm}{dt}\dot{x}^\alpha \tag{4}$$

$$\frac{d\dot{x}^\alpha}{dt} + \underset{(g)}{\Gamma^\alpha_{\cdot\beta\gamma}}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = \frac{1}{m}\frac{\partial U}{\partial x^k} g^{\alpha k} - \frac{1}{2m}\frac{dm}{dt}\dot{x}^\alpha \tag{5}$$

The total derivative of the kinetic energy

$$T = \frac{1}{2}\, \mathrm{m} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \tag{6}$$

is given by

$$\frac{dT}{dt} = \frac{1}{2}\frac{dm}{dt} g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + m\frac{\delta \dot{x}_\alpha}{\delta t}\dot{x}^\alpha$$

or, in view of (4), by

$$\frac{dT}{dt} = \frac{\partial U}{\partial x^\alpha}\dot{x}^\alpha$$

From this we obtain the energy integral

$$T - U = T + V = h = \mathrm{const} \tag{7}$$

This is satisfied also by the corresponding dynamically variable system.

3. In the space $V_n^{\,0}$ with the metric

$$d\sigma^2 = 2(h - V)\, ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta \tag{8}$$

which is conformal to the space V_n with the metric (1), the considered motion takes place along geodetic lines. In order to show this, we transform (5) from V_n into $V_n^{\,\circ}$. For this it is sufficient to transform

$$\underset{(g)}{\Gamma^\alpha_{\cdot\beta\gamma}} = \frac{1}{2} g^{\alpha\lambda}(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) \tag{9}$$

where $\partial_\alpha = \partial/\partial x^\alpha$ is a partial derivative.

From (7) and (1) we can obtain

$$a_{\alpha\beta} = 2(h - V)\, g_{\alpha\beta}, \qquad g_{\alpha\beta} = \frac{a_{\alpha\beta}}{2(h - V)}$$
$$a^{\alpha\beta} = \frac{g^{\alpha\beta}}{2(h - V)}, \qquad g^{\alpha\beta} = 2(h - V)\, a^{\alpha\beta} \tag{10}$$

Because of these relations, (9) can be transformed into

$$\underset{(g)}{\Gamma^{\alpha}_{\cdot\beta\gamma}} = \underset{(a)}{\Gamma^{\alpha}_{\cdot\beta\gamma}} + \frac{1}{2(h - V)}(\delta_{\gamma}{}^{\alpha}\partial_{\beta}V + \delta_{\beta}{}^{\alpha}\partial_{\gamma}V - \partial_{\lambda}V a_{\beta\gamma}a^{\alpha\lambda})$$

Substituting this into (5) we obtain, because of (1), (8) and (10)

$$\frac{d\dot{x}^{\alpha}}{dt} + \underset{(a)}{\Gamma^{\alpha}_{\cdot\beta\gamma}}\frac{dx^{\beta}}{dt}\frac{dx^{\gamma}}{dt} = -\frac{1}{h - V}\dot{x}^{\alpha}\frac{dV}{dt}\frac{dx^{\alpha}}{dt} - \frac{1}{2m}\frac{dm}{dt}\frac{dx^{\alpha}}{dt} \tag{11}$$

which are the final equations of motion of the point in V_n°.

On the other hand, the equation of the geodesic in V_n° is

$$\frac{d\dot{x}^{\alpha}}{dt} + \underset{(a)}{\Gamma^{\alpha}_{\cdot\beta\gamma}}\frac{dx^{\beta}}{dt}\frac{dx^{\gamma}}{dt} = \frac{\ddot{\sigma}}{\dot{\sigma}}\frac{dx^{\alpha}}{dt} \qquad \left(\dot{\sigma} = \frac{d\sigma}{dt}\right)$$

In view of (8), (6) and (7), the right-hand side of the last equation is equal to

$$\frac{\ddot{\sigma}}{\dot{\sigma}}\frac{dx^{\alpha}}{dt} = -\frac{1}{h - V}\frac{dV}{dt}\frac{dx^{\alpha}}{dt} - \frac{1}{2m}\frac{dm}{dt}\frac{dx^{\alpha}}{dt} \tag{12}$$

From this and Equation (11) we obtain the result: a dynamically variable point in a conservative field moves along the geodesics

$$\frac{d^2x^{\alpha}}{d\sigma^2} + \Gamma^{\alpha}_{\cdot\beta\gamma}\frac{dx^{\beta}}{d\sigma}\frac{dx^{\gamma}}{d\sigma} = 0 \tag{13}$$

if the velocity of the particles is colinear and equal to half the velocity of the point.

4. It has been shown that in this case the Maupertuis-Lagrange principle applies.

BIBLIOGRAPHY

1. Bilimovih, B.A., O jednom opshtem differentsiijal'nom printsipu (On a general differential principle)(In Serbian). Belgrade, 1959.

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